

SYMMETRIC PRODUCTS OF A REAL CURVE AND THE MODULI SPACE OF HIGGS BUNDLES

THOMAS JOHN BAIRD

ABSTRACT. Consider a Riemann surface X of genus $g \geq 2$ equipped with an antiholomorphic involution τ . This induces a natural involution on the moduli space $M(r, d)$ of semistable Higgs bundles of rank r and degree d . If D is a divisor such that $\tau(D) = D$, this restricts to an involution on the moduli space $M(r, D)$ of those Higgs bundles with fixed determinant $\mathcal{O}(D)$ and trace-free Higgs field. The fixed point sets of these involutions $M(r, d)^\tau$ and $M(r, D)^\tau$ are (A, A, B) -branes introduced by Baraglia-Schaposnik [4]. In this paper, we derive formulas for the mod 2 Betti numbers of $M(r, d)^\tau$ and $M(r, D)^\tau$ when $r = 2$ and d is odd. In the course of this calculation, we also compute the mod 2 cohomology ring of $SP^m(X)^\tau$, the fixed point set of the involution induced by τ on symmetric products of the Riemann surface.

1. INTRODUCTION

Let X denote a Riemann surface of genus $g \geq 2$ with canonical bundle K . A *Higgs bundle* (E, Φ) over X consists of a holomorphic vector bundle E over X and a section $\Phi \in H^0(X, \text{Hom}(E, E \otimes K))$ called the *Higgs field*. A Higgs field is called *stable* if all proper vector subbundles $F \subset E$ such that $\Phi(F) \subseteq F \otimes K$ satisfy $\deg(F)/\text{rank}(F) \leq \deg(E)/\text{rank}(E)$. In [18], Hitchin constructed the moduli space $M(r, d)$ of semistable Higgs bundles. We will always assume that r and d are coprime, so $M(r, d)$ is non-singular.

Fix a divisor $D \in \text{Div}(X)$. Define $M(r, D)$ to be the subvariety of $M(r, d)$ of Higgs bundles (E, Φ) for which $\wedge^r E \cong \mathcal{O}(D)$ and $\text{tr}(\Phi) = 0$. Both $M(r, d)$ and $M(r, D)$ admit a complete hyperkähler metric: a Riemannian metric g which is Kähler with respect to three different complex structures I, J, K that satisfy the quaternionic relations. We denote by $\omega_I, \omega_J, \omega_K$ the corresponding Kähler forms.

Suppose that X admits an anti-holomorphic involution τ . We call (X, τ) a *real curve*. This induces an involution on $M(r, d)$ (which we also call τ) sending a pair (E, Φ) to $\tau(E, \Phi) := (\tau(E), \tau(\Phi))$, where $\tau(E) = \tau^* \overline{E}$ is the conjugate pull-back and $\tau(\Phi)$ is the composition

$$\tau^* \overline{E} \xrightarrow{(\tau^*)^{-1}} E \xrightarrow{\Phi} E \otimes K \xrightarrow{\tau^*} \tau^*(\overline{E} \otimes \overline{K}) \xrightarrow{\cong} \tau^*(\overline{E}) \otimes K$$

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where we have used the natural isomorphism $K \cong \tau^* \overline{K}$ determined by the fact that τ is anti-holomorphic. If $D \in \text{Div}(X)$ is a *real divisor* in the sense that $\tau(D) = D$, then τ restricts to an involution of $M(r, D)$.

This involution was considered by Baraglia-Schaposnik [4]. It preserves the hyperkähler metric, is anti-holomorphic with respect to I, J and is holomorphic with respect to K . Consequently, the fixed point sets of the involutions $M(r, d)^\tau$ and $M(r, D)^\tau$ are real and Lagrangian with respect to I, J and are complex and symplectic with respect to K . Such a submanifold is called an (A, A, B) -brane, and plays a role in the Kapustin-Witten approach to geometric Langlands [20, 5]. In this paper, we derive formulas computing the mod 2 Betti numbers of $M(r, d)^\tau$ and $M(r, D)^\tau$ in case rank $r = 2$ and degree d is odd.

1.1. Outline of the proof. There is natural \mathbb{C}^* -action on $M(r, D)$ by scaling the Higgs field. Hitchin observed that the restricted $U(1)$ -action is Hamiltonian with respect to the symplectic structure ω_I , with proper moment map $\mu : M(r, D) \rightarrow \mathbb{R}$,

$$\mu(E, \Phi) = \|\Phi\|_{L^2}^2.$$

Therefore, by a theorem of Frankel [14], the function μ is a perfect Morse Bott function with respect to rational coefficients and the critical points of μ coincide with the $U(1)$ -fixed points. This means we have an equality

$$P_t^{\mathbb{Q}}(M(r, D)) = \sum_{F \text{ component of } M(r, D)^{U(1)}} P_t^{\mathbb{Q}}(F) t^{2d_F}$$

where $P_t^{\mathbb{K}}(Y) := \sum_{i=0}^{\infty} \dim(H^i(Y; \mathbb{K})) t^i$ is the Poincaré series and $2d_F$ is the Morse index of the path component F (which is necessarily even because the negative normal bundles are symplectic). This reduces the calculation of the rational Betti numbers of $M(r, D)$ to calculating the Betti numbers of the fixed point components F and their Morse indices $2d_F$. This was carried out for rank $r = 2$ by Hitchin [18] and for rank $r = 3$ by Gothen [16].

Similar considerations apply to compute mod 2 Betti numbers of $M(r, D)^\tau$. The involution is compatible with the $U(1)$ -action in the sense that $e^{i\theta} \circ \tau = \tau \circ e^{-i\theta}$ and $\mu \circ \tau = \mu$. In this circumstance, a theorem of Duistermaat [12, 6] tells us that the restriction of μ to $M(r, D)^\tau$ is a perfect Morse-Bott function with respect to mod 2 coefficients. The set of critical points of μ restricted to $M(r, D)^\tau$ coincides with $M(r, D)^\tau \cap M(r, D)^{U(1)}$ and the Morse indices are halved (since they compute the dimension of Lagrangian vector subbundles of symplectic vector bundles). Consequently, we obtain the formula

$$(1.1) \quad P_t^{\mathbb{Z}_2}(M(r, D)^\tau) = \sum_{F \text{ component of } M(r, D)^{U(1)}} P_t^{\mathbb{Z}_2}(F^\tau) t^{d_F}.$$

Thus to compute the mod 2 Betti numbers of $M(r, D)^\tau$ it remains only to compute those of F^τ .

The Morse function μ is globally minimized on $M(r, D)$ exactly when the Higgs field vanishes. Therefore the minimizing set of μ on $M(r, D)$ is identified with the *moduli space of stable vector bundles* $N(r, D)$ of rank r and determinant $\mathcal{O}(D)$. The global minimizing set of μ restricted to $M(r, D)^\tau$ is consequently identified with $N(r, D)^\tau$, the *moduli space of Real vector bundles* of rank r and determinant $\mathcal{O}(D)$. This moduli space was introduced in [8, 24] and its mod 2 Betti numbers were computed in [2, 3, 21].

Restrict now to the case where the rank $r = 2$. Hitchin shows that the remaining $U(1)$ -fixed points are represented by pairs (E, Φ) of the form

$$\mathcal{E} = L \oplus (L^* \otimes \mathcal{O}(D)), \quad \Phi = \begin{bmatrix} 0 & 0 \\ \varphi & 0 \end{bmatrix}$$

where $\varphi \in H^0(L^{-2} \otimes K(D))$. The fixed point components are identified with pullbacks of the form

$$(1.2) \quad \begin{array}{ccc} F_l & \longrightarrow & Pic_l(X) \\ \downarrow & & \downarrow sq \\ SP^m(X) & \xrightarrow{aj} & Pic_m(X) \end{array}$$

where $SP^m(X)$ is the m -fold symmetric product of X , aj is the Abel-Jacobi map, sq is the map sending $[L]$ to $[L^{-2} \otimes K(D)]$, $m = 2g - 2 - 2l + d$, and l ranges between $1 \leq l \leq g - 1$. Here, sq is a 2^{2g} -fold covering map that can be identified with the squaring map under an appropriate translation $Pic_l(X) \cong Pic_m(X)$ to the Jacobian $Jac(X) := Pic_0(X)$ which is isomorphic to $U(1)^{2g}$ as a Lie group.

The diagram (1.2) is equivariant with respect to the induced τ -actions and we identify F_l^τ with the pull-back of the restriction to τ -fixed points

$$(1.3) \quad \begin{array}{ccc} F_l^\tau & \longrightarrow & Pic_l(X)^\tau \\ \downarrow & & \downarrow sq \\ SP^m(X)^\tau & \xrightarrow{aj} & Pic_m(X)^\tau. \end{array}$$

Thus we are reduced to computing mod 2 Betti numbers of covering spaces of path components of $SP^m(X)^\tau$. We first compute the mod 2 cohomology ring of path components of $SP^m(X)^\tau$ in §3. We then employ an Eilenberg-Moore spectral sequence argument to compute the Betti numbers of the cover in §4. In §5 we input the result into equation (1.1) to calculate the Betti numbers of $M(2, D)^\tau$, from which the Betti numbers of $M(2, d)^\tau$ are easily determined.

2. TOPOLOGY OF REAL CURVES AND PICARD GROUPS

We summarize some material from Gross-Harris [17]. Given a real curve (X, τ) , the fixed point set X^τ is a union of circles that we call *real circles*. Connected real curves

(X, τ) are classified by three invariants (g, n, a) , where g is the genus, n is the number of real circles, and either $a = 1$ if $X \setminus X^\tau$ has connected, or $a = 0$ if it is disconnected. The range of possible values is

$$\{(g, n, a) \mid 1 - a \leq n \leq g + 1 - a, \text{ and if } a = 0, \text{ then } g - n \equiv 1 \pmod{2}\}.$$

A Real line bundle $(L, \tilde{\tau})$ over (Σ, τ) is a line bundle equipped with an isomorphism $\tilde{\tau} : L \xrightarrow{\sim} \tau^* \overline{L}$. The fixed point set $L^{\tilde{\tau}} \rightarrow \Sigma^\tau$ is an \mathbb{R}^1 -bundle with first Stieffel-Whitney class $w_1(L^\tau) \in H^1(\Sigma^\tau; \mathbb{Z}_2)$. If X^τ is non-empty, then $L \cong \mathcal{O}(D)$ for some real divisor D and we define $w_1(D) = w_1(\mathcal{O}(D)^\tau)$. If $C \subset X^\tau$ is a real circle, then the following are equivalent

- (i) $\mathcal{O}(D)^{\tilde{\tau}}|_C$ is non-orientable,
- (ii) $w_1(D)(C) = 1$,
- (iii) D has odd degree supported on C .

We call C odd with respect to D if it satisfies these equivalent conditions. If k is the number of odd circles of D then

$$(2.1) \quad \deg(D) \equiv w_1(D)(\Sigma^\tau) \equiv k \pmod{2}.$$

Assume now that (X, τ) has $n \geq 1$ real circles. Then $Pic_0(X)^\tau \cong (\mathbb{Z}/2)^{n-1} \times (S^1)^g$ as a Lie group. For general $m \in \mathbb{Z}$, $Pic_m(X)^\tau$ is a torsor for $Pic_0(X)^\tau$ yielding a diffeomorphism $Pic_m(X)^\tau \cong (\mathbb{Z}/2)^{n-1} \times (S^1)^g$. The path components are classified by $w_1([D]) = w_1(D)$ which must satisfy $w_1([D])(X^\tau) \equiv m \pmod{2}$. The Abel-Jacobi map restricts to a map

$$aj : SP^m(X)^\tau \rightarrow Pic_m(X)^\tau$$

which is injective on π_0 and surjective on π_0 if $m \geq n - 1$.

If $X^\tau = \emptyset$, then the behaviour of $Pic_m(X)^\tau$ is complicated by the existence of *quaternionic line bundles* [8]. In particular, there exist divisor classes $[D]$ that are fixed by τ but are not represented by a real divisor. This issue is unimportant for our purposes, because our condition that $\deg(D)$ is odd will force (X, τ) to have real points by (2.1).

3. SYMMETRIC PRODUCTS OF A REAL CURVE

Let (X, τ) be a real curve of genus g . For $m \geq 0$, the symmetric product $SP^m(X) := (X \times \dots \times X)/S_m$ is a m -dimensional complex manifold, which is identified with the set of effective divisors of degree d (by convention $SP^0(X)$ is a point). The involution τ acts on $SP^m(X)$ and our goal in this section is to calculate the mod 2 cohomology ring of the path components of the fixed point set $SP^m(X)^\tau$.

3.1. Symmetric products and the case $X^\tau = \emptyset$. We review the cohomology theory of symmetric products of surfaces and prove a few results for later application. See [9] for a nice survey of this topic. All cohomology is taken with coefficients \mathbb{Z}_2 unless otherwise indicated.

Consider the sequence of inclusions

$$X = SP^1(X) \hookrightarrow SP^2(X) \hookrightarrow \dots$$

defined by adding a basepoint at each step. The colimit of this sequence, denoted $SP^\infty(X)$, is the free abelian monoid on X . The Dold-Thom Theorem yields a homotopy equivalence

$$SP^\infty(X) \sim K(H^1(X; \mathbb{Z}), 1) \times K(H^2(X; \mathbb{Z}), 2) \sim Jac(X) \times \mathbb{C}P^\infty$$

where $Jac(X)$ is the Jacobian of X . The cohomology ring is a graded symmetric algebra

$$H^*(SP^\infty(X)) \cong \wedge(x_1, \dots, x_g, y_1, \dots, y_g) \otimes S(\eta)$$

where $x_1, \dots, x_g, y_1, \dots, y_g$ have degree one and η has degree two. The following formulas hold over \mathbb{Z} , but we work always over \mathbb{Z}_2 . MacDonald [M] proved:

Theorem 3.1. *The inclusion $SP^m(X) \hookrightarrow SP^\infty(X)$ induces a surjection in cohomology*

$$\wedge(x_1, \dots, x_g, y_1, \dots, y_g) \otimes S(\eta) \rightarrow H^*(SP^m(X))$$

with relations generated by

$$(3.1) \quad x_{i_1} \dots x_{i_a} y_{j_1} \dots y_{j_b} (x_{k_1} y_{k_1} - \eta) \dots (x_{k_c} y_{k_c} - \eta) \eta^q = 0.$$

for all choices $i_1, \dots, i_a, j_1, \dots, j_b, k_1, \dots, k_c$ of distinct integers from 1 to g inclusive such that $a + b + 2c + q = m + 1$. The \mathbb{Z}_2 -Poincaré series and Euler characteristic statisfy

$$P_t(SP^m(X)) = \sum_{i=0}^{\min(2g, m)} \binom{2g}{i} (t^i + t^{i+2} + \dots + t^{2m-i}) = \sum_{i=0}^{\min(2g, m)} \binom{2g}{i} \frac{t^{2m+2-i} - t^i}{t^2 - 1}$$

$$\chi(SP^m(X)) = (-1)^m \binom{2g-2}{m}$$

We will replace $SP^\infty(X)$ with a finite dimensional approximation

$$(3.2) \quad SP^m(X) \rightarrow Pic_m(X) \times \mathbb{C}P^m$$

inducing a surjection on cohomology with the same generators and relations. The first coordinate function of (3.2) is the Abel-Jacobi map, sending D to $[D]$. For the second, define a continuous map

$$(3.3) \quad X \rightarrow S^2 \cong \mathbb{C}P^1$$

by collapsing the complement of a disk in X to a point (obviously this is not holomorphic). The second coordinate function is the induced map $SP^m(X) \rightarrow SP^m(\mathbb{C}P^1) \cong P(\text{Sym}^m(\mathbb{C}^2)) \cong \mathbb{C}P^m$. If X^τ is non-empty, then map (3.2) can be chosen τ -equivariant with respect to the standard involution on $\mathbb{C}P^m$.

If X^τ is empty, then every element of $SP^m(X)^\tau$ must be a sum of conjugate pairs of points. But conjugate pairs of points are in one-to-one correspondence with points in the quotient space X/τ , which is a non-orientable surface homeomorphic to the $(g+1)$ -fold connected sum $N_g := \mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$. Thus,

Proposition 3.2. *Suppose (X, τ) is a real curve of genus g and $X^\tau = \emptyset$. Then*

$$SP^m(X)^\tau \cong \begin{cases} \emptyset & \text{if } m \text{ is odd} \\ SP^{m/2}(X/\tau) = SP^{m/2}(N_g) & \text{if } m \text{ is even.} \end{cases}$$

The cohomology ring of $SP^m(N_g)$ was computed by Kallel and Salvatore [19].

Theorem 3.3. *The inclusion $SP^m(N_g) \hookrightarrow SP^\infty(N_g) \sim K(H_1(N_g; \mathbb{Z}), 1) \sim (S^1)^g \times \mathbb{R}P^\infty$ induces a surjection in cohomology*

$$\wedge(h_1, \dots, h_g) \otimes S(w) \rightarrow H^*(SP^m(N_g); \mathbb{Z}_2)$$

where h_1, \dots, h_g, w all have degree one. Let $f_i = h_i + w$ for $i = 1, \dots, g$, let $f_{g+1} = w$ and $b = w^2$. Then the relations are generated by

$$(3.4) \quad f_{i_1} \dots f_{i_r} b^t = 0.$$

for all choices i_1, \dots, i_r of distinct integers from 1 to $g+1$ inclusive such that $r+t = m+1$. The \mathbb{Z}_2 -Poincaré series and Euler characteristic satisfy

$$P_t(SP^m(N_g)) = \sum_{i=0}^{\min(g,m)} \binom{g}{i} (t^i + t^{i+1} + \dots + t^{2m-i}) = \sum_{i=0}^{\min(g,m)} \binom{g}{i} \frac{t^{2m-i+1} - t^i}{t-1}$$

$$\chi(SP^m(N_g)) = (-1)^m \binom{g-1}{m}$$

Remark 3.5. When $g \geq 1$, one can use the identity $\binom{2g}{i} = \binom{2g-1}{i} + \binom{2g-1}{i-1}$, to get $P_t(SP^m(N_{2g-1})) = P_t(SP^m(\Sigma_g))$. This can also be deduced from the equality $P_t(N_{2g-1}) = P_t(\Sigma_g)$ by MacDonald's formula [23].

It will be useful to have a geometric understanding of the generator b in Theorem 3.3. Let $C \subset N_g$ be a loop with non-orientable tubular neighbourhood U . Taking the quotient of N_g by the complement of U produces a manifold homeomorphic to a real projective plane $N_g/(N_g \setminus U) \cong \mathbb{R}P^2$. Taking symmetric products determines a map $q_m : SP^m(N_g) \rightarrow SP^m(\mathbb{R}P^2) \cong \mathbb{R}P^{2m}$ where the homeomorphism $SP^m(\mathbb{R}P^2) \cong \mathbb{R}P^{2m}$ is due to Dupont-Lusztig [11]. Recall that if $m \geq 1$, then $H^*(\mathbb{R}P^{2m})$ is a truncated polynomial ring generated by an element $w \in H^1(\mathbb{R}P^{2m})$.

Lemma 3.4. *In terms of the presentation in Theorem 3.3, $b \in H^2(SP^m(N_g))$ is the image of $w^2 \in H^*(\mathbb{R}P^{2m})$ under q_m^* .*

Proof. This is trivially true if $m = 0$, so assume $m \geq 1$. It is easy to check that $q_m : SP^m(N_g) \rightarrow SP^m(\mathbb{R}P^2)$ is a map of degree one, so by Poincaré duality it induces an injection in cohomology (remember we are working with \mathbb{Z}_2 -coefficients). Thus $q_m^*(w^2) = q_m^*(w)^2$ is a non-zero square in $H^2(SP^m(N_g))$. But the only non-zero square in $H^2(SP^m(N_g))$ is b so $q_m^*(w^2) = b$. \square

Suppose that $\Sigma = \Sigma_g$ is a orientable surface, and $N = N_{2g+k-1}$ is the non-orientable surface obtained by performing a real blow up at k different points of Σ . We have a quotient map $bd : N \rightarrow \Sigma$ inducing maps on symmetric products $bd_m : SP^m(N) \rightarrow SP^m(\Sigma)$.

Lemma 3.5. *In terms of the presentations Theorem 3.1 and Theorem 3.3, $bd_m^*(\eta) = b$.*

Proof. This is trivially true if $m = 0$ so assume $m \geq 1$. We can arrange the maps bd and q so that we have a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{bd} & \Sigma \\ \downarrow q & & \downarrow \bar{q} \\ \mathbb{R}P^2 & \xrightarrow{\bar{bd}} & S^2 = \mathbb{C}P^1 \end{array}$$

where all maps are quotient maps, \bar{q} is the quotient map defined (3.3). Taking symmetric products yields

$$\begin{array}{ccc} SP^m(N) & \xrightarrow{bd_m} & SP^m(\Sigma) \\ \downarrow q_m & & \downarrow \bar{q}_m \\ \mathbb{R}P^{2m} \cong SP^m(\mathbb{R}P^2) & \xrightarrow{\bar{bd}_m} & SP^m(S^2) = \mathbb{C}P^m. \end{array}$$

Since bd_m has degree one, it induces an injection on cohomology. By definition, $\eta = \bar{q}^*(c)$ where $c \in H^2(\mathbb{C}P^n)$ is the generator. Therefore $bd_m^*(\eta) = q_m^*(\bar{bd}_m^*(c)) = bd_m^*(w^2) = bd_m^*(w)^2$ is a non-zero square, so it must equal b .

□

Later we will encounter a case where X has two path components that are transposed by τ . The following is proven the same way as Proposition 3.2.

Proposition 3.6. *Suppose that X is a disconnected union of two genus g curves $X = \Sigma_g \coprod \Sigma_g$ and that τ transposes these two path components. Then*

$$SP^m(X)^\tau \cong \begin{cases} \emptyset & \text{if } m \text{ is odd} \\ SP^{m/2}(X/\tau) = SP^{m/2}(\Sigma_g) & \text{if } m \text{ is even.} \end{cases}$$

3.2. Equivariant cohomology and localization. Consider a $\mathbb{Z}/2$ representation on a \mathbb{Z}_2 -vector space V defined by a linear map $\varphi \in GL(V)$ such that $\varphi^2 = Id_V$. The map $Id_V + \varphi : V \rightarrow V$ squares to zero. Let $Z = \ker(Id_V + \varphi) = V^{\mathbb{Z}/2}$ be the fixed point subspace, let $B = \text{im}(Id_V + \varphi)$ and define $V_{triv} := Z/B$. Then V decomposes into a direct sum $\text{rank}(V_{triv})$ many trivial representations of rank one and $\text{rank}(B)$ many indecomposable (not irreducible!) representations of rank two given by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We have natural isomorphisms in group cohomology

$$H^j(\mathbb{Z}/2; V) = \begin{cases} V^{\mathbb{Z}/2} & \text{if } j = 0 \\ V_{triv} & \text{if } j > 0 \end{cases}.$$

As a module over $H^*(\mathbb{Z}/2; \mathbb{Z}_2) = \mathbb{Z}_2[w]$, $H^*(\mathbb{Z}/2; V)$ decomposes as a the direct sum of a free module $V_{triv} \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[w]$ and a torsion module concentrated in degree zero.

Suppose now that $\mathbb{Z}/2$ acts on a space Y , inducing $\mathbb{Z}/2$ -representations $H^j(Y; \mathbb{Z}_2)$. The Serre spectral sequence for the Borel construction converges to $H_{\mathbb{Z}/2}^*(Y; \mathbb{Z}_2)$ and satisfies

$$(3.6) \quad E_2^{i,j} = \begin{cases} H^i(Y; \mathbb{Z}_2)^{\mathbb{Z}/2} & \text{if } j = 0 \\ H^i(Y; \mathbb{Z}_2)_{triv} & \text{if } j > 0. \end{cases}$$

In particular, this means that $E_2^{*,*}$ is isomorphic to $H^*(Y)_{triv} \otimes_{\mathbb{Z}_2} H^*(B\mathbb{Z}/2)$ modulo $H^*(\mathbb{Z}/2)$ -torsion. Combining this with the localization theorem (see [1] §3.1) yields the following.

Proposition 3.7. *Let Y be a compact $\mathbb{Z}/2$ -manifold. Then*

$$\dim(H^*(Y; \mathbb{Z}_2)_{triv}) \geq \dim(H^*(Y^{\mathbb{Z}/2}; \mathbb{Z}_2))$$

with equality if and only if the Serre spectral sequence of the Borel construction stabilizes on page $E_2^{,*}$. We call Y “weakly tight” when equality holds.*

Proof. This is an easy consequence of the localization theorem (and is proven in the special case when $\mathbb{Z}/2$ acts trivially on $H^*(Y)$ in [1] Theorem 3.10.4). Let Q be the quotient field of $H^*(B\mathbb{Z}/2) = H$. Then we have an isomorphism

$$H_{\mathbb{Z}/2}^*(Y) \otimes_H Q \cong H_{\mathbb{Z}/2}^*(Y^{\mathbb{Z}/2}) \otimes_H Q = H^*(Y^{\mathbb{Z}/2}) \otimes_{\mathbb{Z}_2} Q$$

which has the same dimension as $H^*(Y^{\mathbb{Z}/2})$. On the other hand, there is a spectral sequence

$$E_2^{*,*} \otimes_H Q = H^*(Y)_{triv} \otimes_{\mathbb{Z}_2} Q \Rightarrow H_{\mathbb{Z}/2}^*(Y) \otimes_H Q.$$

The result follows. □

Given a real curve (X, τ) and $m \geq 0$ there is an induced involution on $Pic_m(X) \times \mathbb{C}P^m$ sending $([D], [v]) \rightarrow ([\tau(D)], [\bar{v}])$.

Proposition 3.8. *Suppose X^τ is non-empty. Then $Pic_m(X) \times \mathbb{C}P^m$ is weakly tight with respect to the induced involution.*

Proof. Suppose that X^τ consists of $n \geq 1$ real circles and define $b = n - 1$. The induced action of τ_* on $H^1(X) \cong H^1(Pic_m(X))$ is described in Gross-Harris ([17] §4). In particular, $Id + \tau^*$ has rank $g + 1 - n$, so $H^1(Pic_m(X))_{triv}$ has dimension $2n - 2 = 2b$. It follows that we can choose a (not necessarily symplectic) basis of $H^1(Pic_m(X))$ so that

$$H^1(Pic_m(X)) = \mathbb{Z}_2\{x_1, \dots, x_{2b}, y_1, z_1, \dots, y_{g-b}, z_{g-b}\}$$

where the involution τ^* fixes x_1, \dots, x_{2b} and transposes y_i and z_i for $i \in 1, \dots, g - b$.

It follows that

$$H^*(Pic_m(X) \times \mathbb{C}P^m)_{triv} \cong \wedge(x_1, \dots, x_{2b}, d_1, \dots, d_{g-b}) \otimes S(c)$$

where $d_i = y_i z_i$ for $i = 1, \dots, g - b$. In particular $\dim(H^*(Pic_m(X) \times \mathbb{C}P^m)_{triv}) = 2^{g+b}(1 + m)$. Meanwhile, the fixed point set satisfies

$$(Pic_m(X) \times \mathbb{C}P^m)^\tau = Pic_m(X)^\tau \times \mathbb{R}P^m \cong \coprod_{2^b} (S^1)^g \times \mathbb{R}P^m$$

so $\dim(H^*((Pic_m(X) \times \mathbb{C}P^m)^\tau)) = 2^{g+b}(1 + m)$ and the result follows. \square

Proposition 3.9. *Suppose that $X^\tau \neq \emptyset$. Then $SP^m(X)$ is weakly tight and the τ -equivariant map (3.2) induces a surjection in equivariant cohomology*

$$H_{\mathbb{Z}/2}^*(Pic_m(X) \times \mathbb{C}P^m; \mathbb{Z}_2) \twoheadrightarrow H_{\mathbb{Z}/2}^*(SP^m(X); \mathbb{Z}_2).$$

Proof. Consider the corresponding morphism in non-equivariant cohomology

$$\varphi^* : H^*(Pic_m(X) \times \mathbb{C}P^m; \mathbb{Z}_2) \rightarrow H^*(SP^m(X); \mathbb{Z}_2).$$

The Macdonald relations (3.1) occur in degree $m + 1$ or higher, so φ^* is an isomorphism in degrees less than $m + 1$. Thus the induced map $\varphi_{triv}^* : H^*(Pic_m(X) \times \mathbb{C}P^m)_{triv} \cong H^*(SP^m(X))_{triv}$ is also an isomorphism in degree less than $m + 1$.

The mod 2 reduction of the standard Kähler class on $Pic_m(X) \times \mathbb{C}P^m$ defined (using the earlier notation of Theorem 3.1) by $\omega = \eta + \sum_{i=1}^g x_i y_i \in H^2(Pic_m(X) \times \mathbb{C}P^m)$ is fixed by τ . The mod 2 reduction of the Lefschetz maps $H^j(SP_m(X)) \rightarrow H^{2m-i}(SP_m(X))$ induced by the image of $\varphi^*(\omega) \in H^2(SP^m(X))^{\mathbb{Z}/2}$ are isomorphisms for all $j \leq m$ yielding $\mathbb{Z}/2$ -equivariant commutative diagrams

$$\begin{array}{ccc} H^j(Pic_m(X) \times \mathbb{C}P^m) & \xrightarrow{\omega^{m-j} \cup} & H^{2m-j}(Pic_m(X) \times \mathbb{C}P^m) \\ \downarrow \cong & & \downarrow \\ H^j(SP^m(X)) & \xrightarrow[\varphi^*(\omega)^{m-j} \cup]{\cong} & H^{2m-j}(SP^m(X)) \end{array}$$

We deduce surjections

$$\begin{aligned} H^{2m-j}(Pic_m(X) \times \mathbb{C}P^m)^{\mathbb{Z}/2} &\twoheadrightarrow H^{2m-j}(SP^m(X))^{\mathbb{Z}/2}. \\ H^{2m-j}(Pic_m(X) \times \mathbb{C}P^m)_{triv} &\twoheadrightarrow H^{2m-j}(SP^m(X))_{triv}. \end{aligned}$$

Given Proposition 3.6, this implies that (3.2) induces a surjection on page 2 of the Serre spectral sequences. Since the spectral sequence for $Pic_m(X) \times \mathbb{C}P^m$ collapses, it must also collapse for $SP^m(X)$ and the result follows. \square

Remark 3.7. It follows that $\dim(H^*(SP^m(X)^\tau)) = \dim(H^*(SP^m(X)))$ if and only if X^τ is a union of $g + 1$ real circles, the maximum allowed by Harnack's inequality. In the language of Biswas-D'Mello [7], this means that $SP^m(X)$ is an M-variety if and only if

X is an M-curve. The if direction was proven in [7] when $m = 2, 3$ or $m \geq 2g - 1$, and extended to all g by Franz [14].

Proposition 3.10. *The restriction of (3.2) to fixed points determines a cohomology surjection*

$$(3.8) \quad H^*(Pic_m(X)^\tau \times \mathbb{R}P^m; \mathbb{Z}_2) \rightarrow H^*(SP^m(X)^\tau; \mathbb{Z}_2).$$

Proof. Let Q be the quotient field of $H := H^*(B\mathbb{Z}/2)$. We have a commutative diagram

$$\begin{array}{ccc} H_{\mathbb{Z}/2}^*(Pic_m(X) \times \mathbb{C}P^m) \otimes_H Q & \longrightarrow & H_{\mathbb{Z}/2}^*(SP^m(X)) \otimes_H Q \\ \downarrow \cong & & \downarrow \cong \\ H^*(Pic_m(X)^\tau \times \mathbb{R}P^m) \otimes_{\mathbb{Z}_2} Q & \longrightarrow & H^*(SP^m(X)^\tau) \otimes_{\mathbb{Z}_2} Q \end{array}$$

where the vertical isomorphisms are from the localization theorem. The top arrow is a surjection by Proposition 3.9, so the bottom arrow is as well. But the bottom arrow is just an extension of scalars of the map (3.8) so it must also be a surjection. \square

Restricting the surjection (3.8) to path components yields the following corollary.

Corollary 3.11. *For every path component $P \subseteq SP^m(X)^\tau$, there exists a surjective ring homomorphism*

$$\wedge(h_1, \dots, h_g) \otimes S(w) \twoheadrightarrow H^*(P)$$

where h_1, \dots, h_g, w each have degree one (compare Theorem 3.3).

Pursuing these methods a bit further, it is possible to derive a formula for the Poincaré polynomial of $SP^m(X)^\tau$. However, the use of spectral sequences obscures the multiplicative structure of the cohomology ring, so we instead take a different approach.

3.3. Path components and cup products. As explained in §2, the path components of $SP^m(X)^\tau$ are classified by Stieffel-Whitney classes $w_1(D)$ which assigns a value 0 or 1 to each real circle. Given $k, s \geq 0$ such that $k + 2s = m$, denote by $P_X(k, s) \subseteq SP^{k+2s}(X)^\tau$ a path component for which k of the real circles are sent to 1. We call these the *odd circles*.

Lemma 3.12. *$P_X(0, 1) \cong N_g$ is a non-orientable surface with orientation double cover homeomorphic to X .*

Proof. Clearly $P_X(0, 1)$ is a 2-manifold without boundary. If X^τ is empty then $P_X(0, 1)$ is naturally identified with $X/\tau \cong N_g$ (Proposition 3.2). Otherwise X/τ is a surface with one boundary component for each real circle in X^τ and $P_X(0, 1)$ can be constructed by attaching a copy of $SP^2(S^1)$ to each boundary component. Since $SP^2(S^1)$ is a Mobius band, we see that $P_X(0, 1)$ is a non-orientable surface without boundary. Its orientation double cover is obtained by taking X and fattening up each real circle to a cylinder, producing a surface homeomorphic to X . \square

Proposition 3.13. *The forgetful map $f : SP^s(P_X(0, 1)) \rightarrow P_X(0, s)$ induces an isomorphism in \mathbb{Z}_2 -cohomology. In particular, there exists a ring isomorphism*

$$H^*(P_X(0, s)) \cong H^*(SP^s(N_g)).$$

Proof. It is easy to see that f has degree one, so by Poincaré duality f^* is injective. Since both rings are generated by a basis of $g + 1$ elements in degree one (Corollary 3.11), f^* must also be surjective. \square

Remark 3.9. I suspect that f is in fact a homotopy equivalence, but I have not been able to prove it.

Next, we consider the general case $P_X(k, s)$. Given (X, τ) a real curve of genus g and select k odd real circles S_1^1, \dots, S_k^1 . Let \overline{X} be the singular surface defined by collapsing each of the S_i^k to distinct point $*_i$. Let X' be the non-singular surface obtained by “normalizing” \overline{X} which has the effect of replacing each collapse point $*_i$ by a pair of points S_i^0 . This X' is a surface of genus $g - k$ if the odd circles are non-separating, and a disconnected union of two surfaces of genus $(g - k + 1)/2$ if the odd circles are separating. We have a diagram

$$(3.10) \quad \begin{array}{ccc} X & & X' \\ & \searrow & \swarrow \\ & \overline{X} & \end{array}$$

restricting to diffeomorphisms

$$(3.11) \quad X \setminus \coprod_{i=1}^k S_i^1 \cong \overline{X} \setminus \coprod_{i=1}^k *_i \cong X' \setminus \coprod_{i=1}^k S_i^0.$$

There are unique involutions $\bar{\tau}, \tau'$ on \overline{X}, X' respectively making (3.10) equivariant. For $0 \leq k' \leq k$ define $P_{\overline{X}}(k', s)$ to be the image of $P_X(k', s)$ under the induced map $SP^m(X) \rightarrow SP^m(\overline{X})$.

Lemma 3.14. *There is natural isomorphism $P_{\overline{X}}(0, s) \cong P_{X'}(0, s)$.*

Proof. $P_{\overline{X}}(0, s)$ consists of $\bar{\tau}$ -symmetric sums of points in \overline{X} for which there are an even number on each real circle and on each $*_i$. Such a sum corresponds to an element in $P_{X'}(0, s)$ which simply divides the points on $*_i$ evenly between the two components of S_i^0 . \square

Consider the map

$$(3.12) \quad \pi : P_X(k, s) \rightarrow P_{\overline{X}}(0, s) \cong P_{X'}(0, s)$$

obtained by projecting $P_X(k, s) \rightarrow P_{\overline{X}}(k, s)$ then subtracting one point from each $*_i$.

Proposition 3.15. *The map (3.12) is a quasi-fibration, with homotopy fibre $P_X(k, 0) \cong (S^1)^k$.*

Proof. We refer to Hatcher ([18] Lemma 4K.3) for the basic properties of quasi-fibrations. Define B_q to be the subset of $P_{\overline{X}}(0, s)$ consisting of sums for which at least $2q$ points lie on $\coprod_{i=1}^k *_i$. Thus

$$\emptyset = B_{s+1} \subseteq B_s \subseteq B_{s-1} \subseteq \dots \subseteq B_0 = P_{\overline{X}}(0, s)$$

is a filtration by closed subsets. We will prove that $f^{-1}(B_q) \rightarrow B_q$ is a quasi-fibration by induction on $s - q$.

First observe that

$$B_q \setminus B_{q+1} = \coprod_{q_1 + \dots + q_k = q} B_{(q_1, \dots, q_k)}$$

where $B_{(q_1, \dots, q_k)} \subseteq P_{X'}(0, s)$ consists of sums with exactly $2q_i$ points on $*_i$ for each i . There is a natural homeomorphism

$$\pi^{-1}(B_{(q_1, \dots, q_k)}) \cong B_{(q_1, \dots, q_k)} \times SP^{2q_1+1}(S^1) \times \dots \times SP^{2q_k+1}(S^1)$$

so π restricts to a trivial fibre bundle over $B_{(q_1, \dots, q_k)}$. Positive symmetric powers of a S^1 are homotopy equivalent to S^1 , so π restricts to a quasi-fibration over $B_q \setminus B_{q+1}$ with homotopy fibre $(S^1)^k$. In particular, $\pi^{-1}(B_s) \rightarrow B_s$ is a quasi-fibration.

Let $X^\tau \subseteq U \subseteq X$ be a tubular neighbourhood which τ -equivariantly deformation retracts onto X^τ . Define $U_q \subseteq P_{X'}(0, s)$ those sums containing at least $2q$ points inside of U . Then U_q deformation retracts onto B_q lifting to a deformation retraction of $\pi^{-1}(U_q)$ onto $\pi^{-1}(B_q)$ inducing homotopy equivalences on fibres.

Suppose inductively that π is a quasi-fibration over B_q . Then the deformation retraction above implies that π is a quasi-fibration over U_q by ([18] Lemma 4K.3 (c)). Since π also quasifibration over $B_{q-1} \setminus B_q$, π must be a quasi-isomorphism over $B_{q-1} = (U_q \cap B_{q-1}) \cup (B_{q-1} \setminus B_q)$ by ([18] Lemma 4K.3 (a)), completing the induction. \square

Proposition 3.16. *Given any divisor $D_0 \in P_X(0, s)$, the map*

$$(3.13) \quad i : P_X(k, 0) \rightarrow P_X(k, s), \quad D \mapsto D + D_0$$

induces a surjection in \mathbb{Z}_2 -cohomology. In particular, the Serre spectral sequence of (3.12) collapses and we obtain an isomorphism

$$(3.14) \quad H^*(P_X(k, s); \mathbb{Z}_2) \cong H^*((S^1)^k; \mathbb{Z}_2) \otimes H^*(P_{X'}(0, s); \mathbb{Z}_2).$$

of graded $H^(P_{X'}(0, s); \mathbb{Z}_2)$ -modules.*

Set $\bar{g} = g - k$. If the odd circles are not separating then the Poincaré series equals

$$P_t(P_X(k, s)) = (1+t)^k \sum_{i=0}^{\min(s, \bar{g})} \binom{\bar{g}}{i} \frac{t^{2s-i+1} - t^i}{t-1}$$

If the odd circles are separating then

$$P_t(P_X(k, s)) = (1+t)^k \sum_{i=0}^{\min(s, \bar{g}+1)} \binom{\bar{g}+1}{i} \frac{t^{2s-i+2} - t^i}{t^2-1}$$

In either case the Euler characteristic satisfies

$$\chi(P_X(k, s)) = \begin{cases} 0 & \text{if } k > 0 \\ (-1)^s \binom{\bar{g}-1}{s} & \text{if } k = 0 \end{cases}$$

Proof. By Corollary 3.11, $H^*(P_X(k, s))$ is generated in $(g+1)$ elements in degree one, $\alpha_1, \dots, \alpha_{g+1} \in H^1(P_X(k, s))$. Therefore, $\text{im}(i^*)$ is generated by $i^*(\alpha_1), \dots, i^*(\alpha_{g+1}) \in H^1(P_X(k, 0))$. Because $H^*(P_X(k, 0))$ is an exterior algebra generated by k elements in degree one, it follows that i^* is surjective if and only if $H^k(P_X(k, 0)) \subseteq \text{im}(i^*)$.

The inclusion i extends to the forgetful map

$$f : P_X(k, 0) \times P_X(0, s) \rightarrow P_X(k, s).$$

This is a degree one map of closed manifolds sending one fundamental class to the other. In terms of the Kunneth decomposition this means that

$$H^k(P_X(k, 0)) \times H^{2s}(P_X(0, s)) \subseteq \text{im}(f^*).$$

On the other hand, the image of f^* is generated by the image of the generators $\alpha_1, \dots, \alpha_{g+1}$ which have the form

$$f^*(\alpha_i) = i^*(\alpha_i) \otimes 1 + 1 \otimes \gamma_i$$

for some $\gamma_i \in H^*(P_X(0, s))$. Therefore, $\text{im}(f^*) \subseteq \text{im}(i^*) \otimes H^*(P_X(0, s))$. It follows that $\text{im}(i^*)$ must contain $H^k(P_X(k, 0))$ and thus i^* is surjective.

Applying the Leray-Hirsch Theorem to the quasi-fibration of Proposition 3.15 yields the isomorphism (3.14). Input the formulas from Theorem 3.1 and Theorem 3.3 to compute the Poincaré series and Euler characteristic. \square

Remark 3.15. If $\bar{g} = g - k \geq 0$, then $P_t(P_X(k, s))$ is independent of whether or not the odd circles are separating, due to Remark 3.5.

Theorem 3.17. *Let (X, τ) be a real curve of genus g with n real circles, and let $P_X(k, s) \subseteq SP^{k+2s}(X)^\tau$ be a path component with k odd circles. If the odd circles are not separating then*

$$(3.16) \quad H^*(P_X(k, s)) \cong \wedge(x_1, \dots, x_k) \otimes H^*(SP^s(N_{g-k})).$$

If the odd circles are separating, then

$$(3.17) \quad H^*(P_X(k, s)) \cong \wedge(x_1, \dots, x_{k-1}) \otimes H^*(SP^s(\Sigma_{(g-k+1)/2}))[\sqrt{\eta}]$$

where we adjoin a square root of $\eta \in H^(SP^s(\Sigma_{(g-k+1)/2}))$ defined in the presentation (3.1).*

Proof. If $s = 0$, then $P_X(k, s) \cong (S^1)^k$ and the result holds. So assume $s \geq 1$. By Proposition 3.16, we have quasi-fibration sequence $P_X(k, 0) \xrightarrow{i} P_X(k, s) \xrightarrow{\pi} P_{X'}(0, s)$ inducing a cohomology injection π^* and surjection i^* of $H^*(P_X(k, s))$ onto $H^*(P_X(k, 0)) = H^*((S^1)^k)$ which is an exterior algebra on k generators of degree one. Choose $x_1, \dots, x_k \in$

$H^*(P_X(k, s))$ so that $i^*(x_1), \dots, i^*(x_k)$ forms a basis for $H^1(P_X(k, 0))$. Then by the Leray-Hirsch Theorem, we have a surjective ring homomorphism

$$(3.18) \quad S(x_1, \dots, x_k) \otimes H^*(P_{X'}(0, s)) \twoheadrightarrow H^*(P_X(k, s)).$$

Recall from Corollary 3.11 that $H^*(P_X(k, s))$ is generated as a ring by elements of degree one, all but one of which squares to zero. Therefore, we can assume that $x_i^2 = 0$ for $i = 1, \dots, k-1$ and the surjection (3.18) factors through a surjection

$$\wedge(x_1, \dots, x_{k-1}) \otimes S(x_k) \otimes H^*(P_{X'}(0, s)) \twoheadrightarrow H^*(P_X(k, s)).$$

If X^τ is non-separating, then there exists $z \in H^1(P_{X'}(0, s)) = H^1(SP^s(N_{g-k}))$ such that $z^2 \neq 0$. Thus either $x_k^2 = 0$ or we can replace it with $x_k + \pi^*(z)$ which satisfies $(x_k + \pi^*(z))^2 = x_k^2 + \pi^*(z)^2 = 0$ proving (3.16).

If X^τ is separating, then every element of $H^1(P_{X'}(0, s)) = H^1(SP^s(\Sigma_{(g-k+1)/2}))$ squares to zero. It follows that $x_k^2 \neq 0$. Recall that the forgetful map $f : P_X(k, 0) \times P_X(0, s) \rightarrow P_X(k, s)$ has degree one, so it induces a cohomology injection. Decompose $f^*(x_k) = y \otimes 1 + 1 \otimes z$ according to Kunneth formula. Then $0 \neq f^*(x_k^2) = y^2 \otimes 1 + 1 \otimes z^2 = 1 \otimes z^2 \in 1 \otimes H^2(P_X(0, s))$. By Corollary 3.11, $H^2(P_X(0, s))$ contains a unique non-zero square. Thus to prove (3.17), it suffices to show that the map $P_X(0, s) \rightarrow P_{X'}(0, s)$ sends η to this unique non-zero square. By Proposition 3.13 we have a commuting diagram

$$\begin{array}{ccc} H^*(SP^s(P_X(0, 1))) & \xleftarrow{\cong} & H^*(P_X(0, s)) \\ \uparrow bd_s^* & & \uparrow \\ H^*(SP^s(P_{X'}(0, 1))) & \xleftarrow{\cong} & H^*(P_{X'}(0, s)) \end{array}$$

where bd_s is the symmetric power of the blow-down map considered in Lemma 3.5, completing the proof. □

4. REAL ABEL-JACOBI AND COVERING SPACES

Recall from §2 that if Σ^τ is a disjoint union of $n \geq 1$ real circles, then for all $l \in \mathbb{Z}$, $Pic_l(X)^\tau$ is a torsor for $Pic_0(X)^\tau \cong (\mathbb{Z}/2)^{n-1} \times (S^1)^g$. Consider the homomorphism $sq : Pic_0(\Sigma)^\tau \rightarrow Pic_0(\Sigma)^\tau$ sending the divisor class $[D]$ to $sq([D]) = [-2D]$. Then sq maps to the identity component forms a $(\mathbb{Z}/2)^{g+n-1}$ -principal covering over it. More generally, given a fixed divisor D_0 of degree d , the map

$$sq : Pic_l(X)^\tau \rightarrow Pic_{d-2l}(X)^\tau$$

sending $[D]$ to $[D_0 - 2D]$ is a $(\mathbb{Z}/2)^{g+n-1}$ -principal cover over the path component of $Pic_{d-2l}(X)^\tau$ corresponding to $w_1(D_0)$.

Applying this to the pull-back diagram (1.3), we identify the real part of the $U(1)$ -fixed point component F_l^τ with a pull-back of the form

$$(4.1) \quad \begin{array}{ccc} F_l^\tau & \longrightarrow & (\mathbb{Z}/2)^{n-1} \times (S^1)^g \\ \downarrow & & \downarrow sq \\ P_X(k, s) & \xrightarrow{aj} & (S^1)^g \end{array}$$

where k is the number of odd circles of D , $k + 2s = m = \deg(K(D)) - 2l$, $(S^1)^g$ is identified with the path component of $Pic_m(X)^\tau$ containing $[K(D)]$, and sq is the group homomorphism sending each element to its square. Clearly then, F_l^τ is a coproduct of 2^{n-1} copies of the pullback

$$(4.2) \quad \begin{array}{ccc} P & \longrightarrow & (S^1)^g \\ \downarrow & & \downarrow sq \\ P_X(k, s) & \xrightarrow{aj} & (S^1)^g. \end{array}$$

We are reduced to computing the mod 2 Betti numbers of P .

In general, suppose $\pi : \tilde{M} \rightarrow M$ is $(\mathbb{Z}/2)^q$ -principal bundle determined by the q -tuple (w_1, \dots, w_q) of Stieffel-Whitney classes $w_i \in H^1(M; \mathbb{Z}_2)$ and M is a finite type CW-complex. The classifying map

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & (S^\infty)^q \\ \downarrow & & \downarrow \\ M & \longrightarrow & (\mathbb{R}P^\infty)^q. \end{array}$$

gives rise to an Eilenberg-Moore Spectral sequence, $EM_*^{*,*}$ converging to $H^*(\tilde{M}; \mathbb{Z}_2)$.¹ The second page $EM_2^{*,*}$ equals the cohomology of a bigraded differential graded algebra

$$(4.3) \quad EM_2^{*,*} = H(\wedge(u_1, \dots, u_k) \otimes H^*(M; \mathbb{Z}_2), \delta)$$

where $\deg(u_i) = (-1, 1)$, $\deg(H^k(M; \mathbb{Z}_2)) = (0, k)$, $\delta(u_i) = w_i$ and $\delta(H^*(M; \mathbb{Z}_2)) = 0$.

In case $q = 1$, $\tilde{M} \rightarrow M$ is a 2-fold cover and (4.3) is equivalent to the associated Thom-Gysin sequence. In particular, because (4.3) has only two non-zero columns when $q = 1$, the spectral sequence collapses to give $P_t(EM_2^{*,*}) = P_t(\tilde{M})$.

Introduce the relation $P_t(X) \geq P_t(Y)$ if the difference $P_t(X) - P_t(Y)$ has only non-negative coefficients.

Lemma 4.1. *Let $\pi : \tilde{M} \rightarrow M$ be a $(\mathbb{Z}/2)^q$ -principal bundle classified by Stieffel-Whitney classes $(w_1, \dots, w_q) \in H^1(M)^{\times q}$. If $w_i^2 = 0$ for all w_i , then $P_t(\tilde{M}) \geq P_t(M)$.*

¹The spectral sequence converges to $H^*(\tilde{M})$ because $\pi_1((\mathbb{R}P^\infty)^q) = (\mathbb{Z}/2)^q$ is a finite 2-group and the homotopy fibre \tilde{M} has finite dimensional cohomology in each degree. See [7] citing [13].

Proof. If $q = 1$ then we have the Thom-Gysin sequence:

$$\dots \rightarrow H^{k-1}(M) \xrightarrow{\varphi_{k-1}} H^k(M) \xrightarrow{\pi^*} H^k(\tilde{M}) \rightarrow H^k(M) \xrightarrow{\varphi_k} H^{k+1}(M) \rightarrow \dots$$

where φ_k is cup product by w_1 . Because $w_1^2 = 0$ we have $\text{im}(\varphi_{k-1}) \subseteq \ker(\varphi_k)$. The result follows by exactness.

For $q > 1$, we construct a sequence of 2-fold covers $\tilde{M} = M_q \rightarrow M_{q-1} \rightarrow \dots \rightarrow M_0 \rightarrow M$ by taking partial quotients by subgroups of $(\mathbb{Z}/2)^q$. The Stieffel-Whitney class for the covering $M_{i+1} \rightarrow M_i$ is the pull-back of $w_i \in H^*(M; \mathbb{Z}_2)$ so it must square to zero. By induction $P_t(\tilde{M}) \geq P_t(M)$. \square

Lemma 4.2. *Let $\tilde{M} \rightarrow M$ be as in Lemma 4.1. Suppose that there is a ring factorization*

$$H^*(M; \mathbb{Z}_2) \cong \wedge(w_1, \dots, w_q) \otimes R$$

for some graded subring $R \leq H^(M; \mathbb{Z}_2)$. Then $P_t(\tilde{M}) = P_t(M)$.*

Proof. By Lemma 4.1 we have $P_t(\tilde{M}) \geq P_t(M)$. For the converse inequality, check that $EM_2^{*,*} \cong \wedge(y_1, \dots, y_q) \otimes R$ where y_i is the class represented by the cocycle $u_i w_i$. Therefore $P_t(M) = P_t(EM_2^{*,*}) \geq P_t(EM_\infty^{*,*}) = P_t(\tilde{M})$. \square

Proposition 4.3. *Consider the covering space $P \rightarrow P_X(k, s)$ defined by the pull-back diagram (4.2). Introduce $\bar{g} = g - k$. Then P has Poincaré series divisible by $(1+t)^k$. If the odd circles are not separating then*

$$\begin{aligned} P_t(P)/(1+t)^k &= P_t(P_{X'}(0, s)) + (-1)^s(2^{\bar{g}} - 1)\chi(P_{X'}(0, s))t^s \\ &= \left[\sum_{i=0}^{\min(s, \bar{g})} \binom{\bar{g}}{i} \frac{t^{2s-i+1} - t^i}{t-1} \right] + (2^{\bar{g}} - 1) \binom{\bar{g}-1}{s} t^s. \end{aligned}$$

If the odd circles are separating then

$$\begin{aligned} P_t(P)/(1+t)^k &= P_t(P_{X'}(0, s)) + (-1)^s(2^{\bar{g}+1} - 1)\chi(P_{X'}(0, s))t^s \\ &= \left[\sum_{i=0}^{\min(s, \bar{g}+1)} \binom{\bar{g}+1}{i} \frac{t^{2s-i+2} - t^i}{t^2-1} \right] + (2^{\bar{g}+1} - 1) \binom{\bar{g}-1}{s} t^s. \end{aligned}$$

Proof. The $(\mathbb{Z}/2)^g$ -principal bundle $P \rightarrow P_X(k, s)$ has Stieffel-Whitney classes (w_1, \dots, w_g) given by the image of a basis $z_1, \dots, z_g \in H^1((S^1)^g)$ under

$$aj_1^* : H^1((S^1)^g) \rightarrow H^1(P_X(k, s)).$$

Let $q := \bar{g} + 1$ if X^τ is separating, and $q := \bar{g}$ if not.

If $s = 0$, then $P_X(k, 0) = (S^1)^k$ and by Theorem 3.17 we may choose the basis so that $w_1 = \dots = w_q = 0$ and $w_{q+1}, \dots, w_g \in H^1(P_X(k, 0))$ are linearly independent. It follows that

$$P = \coprod_{2^q} (S^1)^k$$

in accord with formulas above.

Now suppose that $s \geq 1$. Then any basis z_1, \dots, z_g will map to a linearly independent set w_1, \dots, w_g . We can choose the basis so that w_1, \dots, w_q lie in the image of π^* where $\pi : P_X(k, s) \rightarrow P_{X'}(0, s)$ is the quasi-fibration (3.12), and w_{q+1}, \dots, w_g are sent to linearly independent elements in the cohomology of $P_X(k, 0) \cong (S^1)^k$, the quasi-fibre of π .

Consider the intermediate covering space

$$P \rightarrow P' \rightarrow P_X(k, s)$$

where P' has is the $(\mathbb{Z}/2)^q$ -bundle classified by w_1, \dots, w_q . Then P' is a pull-back

$$\begin{array}{ccc} P' & \longrightarrow & P_X(k, s) \\ \downarrow & & \downarrow \\ P'' & \longrightarrow & P_{X'}(0, s) \end{array}$$

In particular, $P' \rightarrow P''$ is a quasi-fibration with quasi-fibre $(S^1)^k$, whose Serre spectral sequence collapses because it is a pull-back of (3.12), so $P_t(P') = (1+t)^k P_t(P'')$.

The $(\mathbb{Z}/2)^{g-q}$ -bundle $P \rightarrow P'$ is classified by the images of w_{q+1}, \dots, w_g . By Theorem 3.17, we see that it satisfies the conditions of Lemma 4.2, so

$$P_t(P) = P_t(P') = (1+t)^k P_t(P'').$$

It remains to compute $P_t(P'')$. By Lemma 4.1, we know $P_t(P'') \geq P_t(P_{X'}(0, s))$.

If X^τ is non-separating then by Proposition 3.13 we have an isomorphism $H^*(P_{X'}(0, s)) \cong H^*(SP^s(N_q))$. If X^τ is separating, then $H^*(P_{X'}(0, s)) \cong H^*(SP^s(\Sigma_{(q/2)}))$. Recall from Corollary 3.11 and Proposition 3.6 we have a surjective homomorphism

$$(4.4) \quad \wedge(x_1, \dots, x_q) \otimes S(\alpha) \rightarrow H^*(P_{X'}(0, s))$$

which is an isomorphism in degree less than $s+1$, where $\alpha = w$ has degree one if X^τ is non-separating and $\alpha = \eta$ has degree two if X^τ is separating. The Stiefel-Whitney invariants of P'' are the images of (x_1, \dots, x_q) so (4.4) extends naturally to a dga morphism

$$\wedge(u_1, \dots, u_q) \otimes \wedge(x_1, \dots, x_q) \otimes S(\alpha) \rightarrow \wedge(u_1, \dots, u_q) \otimes H^*(P_{X'}(0, s)).$$

where $\delta(u_i) = x_i$. This yields a map in cohomology (where $y_i := [u_i x_i]$)

$$\wedge(y_1, \dots, y_q) \otimes S(\alpha) \rightarrow E_2^{*,*}$$

which is an isomorphism total degree less than s . Thus

$$P_t(P_{X'}(0, s)) = P_t(EM_2^{*,*}) \geq P_t(EM_\infty^{*,*}) = P_t(P'') \text{ mod } t^s$$

so $P_{X'}(0, s)$ and P'' have the same Betti numbers for degrees less than s . Since they are both compact manifolds of dimension $2s$, by Poincaré duality they also have the same Betti numbers for degree greater than s , so they differ only in the middle degree s . Because the Euler characteristics satisfy $\chi(P'') = 2^q \chi(P_{X'}(0, s))$, this forces

$$\dim(H^s(P'')) = \dim(H^s(P_{X'}(0, s))) + (-1)^s (2^q - 1) \chi(P_{X'}(0, s)).$$

It only remains to input the formulas from Proposition 3.16. □

5. BETTI NUMBERS OF THE MODULI SPACE

5.1. Betti numbers of $M(2, D)^\tau$.

Theorem 5.1. *Let (Σ, τ) be a real curve of genus $g \geq 2$ with n real circles, let D is a real divisor of odd degree with k odd circles. Let $\bar{g} = g - k$ and $b = n - 1$.*

If the odd circles are not separating then the Poincaré series of $M(2, D)^\tau$ is

$$\begin{aligned} P_t(M(2, D)^\tau) &= \frac{(1+t)^b(1+t^2)^b(1+t^3)^{g-b} - 2^b t^g (1+t)^g}{(1-t)(1-t^2)} \\ &+ 2^b (1+t)^k \sum_{s=0}^{g-(k+3)/2} \left((2^{\bar{g}} - 1) \binom{\bar{g}-1}{s} t^s + \sum_{i=0}^{\min(s, \bar{g})} \binom{\bar{g}}{i} \frac{t^{2s-i+1} - t^i}{t-1} \right) t^{3g-3-k-2s} \end{aligned}$$

If the odd circles are separating, then

$$\begin{aligned} P_t(M(2, D)^\tau) &= \frac{(1+t)^b(1+t^2)^b(1+t^3)^{g-b} - 2^b t^g (1+t)^g}{(1-t)(1-t^2)} \\ &+ 2^b (1+t)^k \sum_{s=0}^{g-(k+3)/2} \left((2^{\bar{g}+1} - 1) \binom{\bar{g}-1}{s} t^s + \sum_{i=0}^{\min(s, \bar{g}+1)} \binom{\bar{g}+1}{i} \frac{t^{2s-i+2} - t^i}{t^2-1} \right) t^{3g-3-k-2s} \end{aligned}$$

Remark 5.1. The range of possible inputs into Theorem 5.1 is as follows (see §2). If the odd circles are not separating, then $k \equiv 1 \pmod{2}$ and either $1 \leq k \leq n \leq g$ or $1 \leq k < n = g + 1$. If the odd circles are separating, then $1 \leq k = n \leq g + 1$ and $g \equiv n + 1 \equiv 0 \pmod{2}$.

Remark 5.2. If $(g, n, k) = (2, 3, 3)$ (so the odd circles must separate), then the second line in the above formula is empty, because $(g - (k + 3)/2) < 0$. In this case, there are no Morse critical points except the global minimum $N(2, D)^\tau$ so the Morse flow deformation retracts $M(2, D)^\tau$ onto $N(2, D)^\tau$, giving $P_t(M(2, D)^\tau) = 1 + 3t + 3t^2 + t^3$.

Proof of Theorem 5.1. Given a real divisor D' , there is a natural isomorphism

$$M(2, D)^\tau \cong M(2, D + 2D')^\tau$$

defined by tensoring by $\mathcal{O}(D')$. Since $\deg(D + 2D') = \deg(D) + 2\deg(D')$ and $w_1(D) = w_1(D + 2D')$, we may assume without loss of generality that $\deg(D) = 1$.

Recall (1.1), adding in Morse indices $d_F = g + 2l - 2$,

$$(5.3) \quad P_t(M(2, D)^\tau) = P_t(N(2, D)^\tau) + \sum_{l=1}^{g-1} P_t(F_l^\tau) t^{g+2l-2}$$

where $N(2, D)^\tau$ is the moduli space of Real bundles of rank 2 and determinant D . By [3] we have

$$P_t(N(2, D)^\tau) = \frac{(1+t)^b(1+t^2)^b(1+t^3)^{g-b} - 2^b t^g (1+t)^g}{(1-t)(1-t^2)}.$$

The remaining terms were calculated in §4. If $m := 2g - 2l - 1 < k$ then $P_l^\tau = \emptyset$. If $m \geq k$ we have

$$P_t(F_l^\tau) = 2^b P_t(P)$$

where P is defined in Proposition 4.3, and $2s := m - k$. We replace the index l by $s = g - l - (k+1)/2$, which ranges from $s = 0$ to $s = g - (k+3)/2$. Inputting into (5.3) completes the proof. \square

Remark 5.4. The Poincaré series $P_t(M(2, D)^\tau)$ is sensitive to whether or not the odd circles are separating. For example, with inputs $(g, n, k) = (2, 1, 1)$ we get

$$P_t(M(2, D)^\tau) = \begin{cases} (1+t)(1+3t^2) & \text{if non-separating} \\ (1+t)(1+5t^2) & \text{if separating} \end{cases}$$

5.2. Betti numbers of $M(2, d)^\tau$. Let (X, τ) be a real curve of genus g with $n \geq 1$ real circles, and consider the moduli of rank r , odd degree d stable Higgs bundles $M(r, d)$ without fixing a determinant ($\gcd(r, d) = 1$ as always). The fixed point set $M(r, d)^\tau$ decomposes into 2^{n-1} path components

$$M(r, d)^\tau = \coprod_w M(r, d)_w^\tau$$

classified by $w \in H^1(X^\tau; \mathbb{Z}_2)$ the first Stiefel-Whitney class of the determinant line bundle, which must satisfy $w(X^\tau) \equiv d \pmod{2}$.

Tensor product defines a natural isomorphism

$$M(r, d) \cong M(r, D) \times_{\Gamma_{2g}} M(1, 0)$$

where $\Gamma_{2g} \cong (\mathbb{Z}/r)^{2g}$ is the r -torsion subgroup of $M(1, 0) = \text{Pic}_0(X) \times H^0(X, K) \cong (S^1)^{2g} \times \mathbb{C}^g$. If we choose a real divisor D such that $w_1(D) = w$, then we similarly obtain an isomorphism

$$M(r, d)_w^\tau \cong M(r, D)^\tau \times_{\Gamma_g} M(1, 0)_0^\tau$$

where $\Gamma_g \cong (\mathbb{Z}/r)^g$ is the r -torsion subgroup of $M(1, 0)_0^\tau \cong (S^1)^g \times \mathbb{R}^g$.

Theorem 5.2. *Let (Σ, τ) be a real curve of genus $g \geq 2$ with n real circles, let D be a real divisor of odd degree d , and let $w = w_1(D)$. Let $\bar{g} = g - k$ and $b = n - 1$.*

If the odd circles are not separating then the mod 2 Poincaré series of $M(2, d)_w^\tau$ is

$$\begin{aligned} P_t(M(2, d)_w^\tau) &= \frac{(1+t)^{b+g}(1+t^2)^b(1+t^3)^{g-b} - 2^b t^g (1+t)^g}{(1-t)(1-t^2)} \\ &+ 2^b (1+t)^{k+g} \sum_{s=0}^{g-(k+3)/2} \left(\sum_{i=0}^{\min(s, \bar{g})} \binom{\bar{g}}{i} \frac{t^{2s-i+1} - t^i}{t-1} \right) t^{3g-3-k-2s} \end{aligned}$$

If the odd circles are separating, then

$$P_t(M(2, d)_w^\tau) = \frac{(1+t)^{b+g}(1+t^2)^b(1+t^3)^{g-b} - 2^b t^g (1+t)^{2g}}{(1-t)(1-t^2)} \\ + 2^b (1+t)^{k+g} \sum_{s=0}^{g-(k+3)/2} \left(\sum_{i=0}^{\min(s, \bar{g}+1)} \binom{\bar{g}+1}{i} \frac{t^{2s-i+2} - t^i}{t^2 - 1} \right) t^{3g-3-k-2s}$$

Proof. The discussion in §1.1 applies analogously to $M(2, d)^\tau$. In particular, the function $\mu : M(2, d) \rightarrow \mathbb{R}$, $\mu(E, \Phi) = \|\Phi\|_{L^2}^2$ restricts to a proper, \mathbb{Z}_2 -perfect Morse-Bott function on $M(2, d)^\tau$, hence also on its path components $M(2, d)_w^\tau$, with critical set equal to $M(2, d)_w^\tau \cap M(2, d)^{U(1)}$. The global minimum of μ on $M(2, d)_w^\tau$ is equal to the moduli space $N(2, d)_w^\tau$ of stable Real vector bundles whose Betti numbers were computed in [2, 21]:

$$P_t(N(2, d)_w^\tau) = (1+t)^g P_t(N(2, D)^\tau).$$

The remaining critical loci have the form $\coprod_{2^b} Q$ where

$$Q := P \times_{\Gamma_g} M(1, 0)_0^\tau$$

and $P \rightarrow P_X(k, s)$ is the Γ_g -principal cover (4.2) for some s . Therefore, projecting onto the first factor determines a fibre bundle

$$(S^1)^g \times \mathbb{R}^g \rightarrow Q \rightarrow P_X(k, s).$$

Since Γ_g acts on $M(1, 0)_0^\tau \cong (S^1)^g \times \mathbb{R}^g$ by translation, $\pi_1(P_X(k, s))$ acts trivially on the cohomology of the fibre, and the Serre spectral sequence gives us an inequality

$$P_t(Q) \leq (1+t)^g P_t(P_X(k, s)).$$

On the other hand, Q is a Γ_g -principal cover of

$$(P/\Gamma_g) \times (M(1, 0)_0^\tau/\Gamma_g) \cong P_X(k, s) \times (S^1)^g \times \mathbb{R}^g$$

determined by Stiffel-Whitney classes that square to zero, so applying Lemma 4.1 we get the converse inequality and conclude that $P_t(Q) = (1+t)^g P_t(P_X(k, s))$. We obtain

$$(5.5) \quad P_t(M(2, d)_w^\tau) = (1+t)^g \left[P_t(N(2, D)^\tau) + \sum_{s=0}^{g-(k+3)/2} 2^b P_t(P_X(k, s)) t^{3g-3-k-2s} \right].$$

□

Corollary 5.3. *The Poincaré series $P_t(M(2, d)^\tau)$ is independent of whether or not the odd circles are separating.*

Proof. If both separating and non-separating examples exist for given (g, n, k) , then $g \geq n = k$, so we can apply Remark 3.15 to (5.5). □

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DEPARTMENT OF MATHEMATICS AND STATISTICS, MEMORIAL UNIVERSITY OF NEWFOUNDLAND,
ST. JOHN’S, NL, CANADA, A1C 5S7, TBAIRD@MUN.CA